Proof of Casteljau method

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Introduction

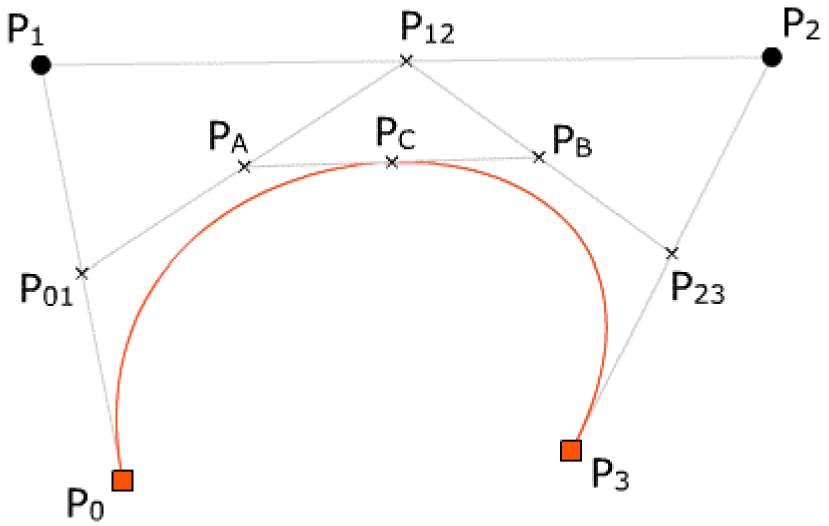
The Casteljau method is a method that shows how to split a cubic bezier curve into two halves cubic bezier curves. Numerous places on the web reference it but I couldn't find a proof and since I don't like taking things for granted, I rewrote my own. The proof is easy, but it is a bit tedious to write.

Hypothesis

Let's consider a bezier curve determined by 4 control points $\{P_0, P_1, P_2, P_3\}$. Let's add to this a few extra points defined as follow:

- P_{01} being the middle of $[P_0, P_1]$
- P_{12} being the middle of $[P_1, P_2]$
- P_{23} being the middle of $[P_2, P_3]$
- P_A being the middle of $[P_{01}, P_{12}]$
- P_B being the middle of $[P_{12}, P_{23}]$
- P_C being the middle of $[P_A, P_B]$

We therefore have a graphic like this:



Goal

We need to prove three things

1. Prove that PC is the middle of the cubic curve, that is B(0.5).
2. Prove that the cubic curve determined by the four control points $\{P_0, P_{01}, P_A, P_C\}$ matches exactly the first half of the main cubic.
3. Prove that the cubic curve determined by the four control points $\{P_C, P_B, P_{23}, P_3\}$ matches exactly the second half of the main cubic.

Notations

Cubic beziers curves are parametric functions where the function for each coordinate (x and y) is the same. In the calculations below, I will therefore simply use the points' names instead of referring to their coordinate x and y for each line.

I will apply this thinking for the function middle (called m) which return the middle of two points with the same function applied to each coordinate:

$m(a,b)=\frac{a+b}{2}$ where a and b are points in a space of any dimension.

Reminder

Also, before we carry on, I want to show quickly the cubic Bezier formula and its form in a easy to use polynomial function.

Remember that the general bezier curve formula with N+1 control points is:

$$\left. B(u)=\sum_{k=0}^{N} P_k \frac{N!}{k!(N-k)!} u^k (1-u)^{N-k} \right\} 0 \le u \le 1$$

A cubic bezier curve has four control points so the formula looks like this:

$$\left. B(u)=\sum_{k=0}^{3} P_k \frac{3!}{k!(3-k)!} u^k (1-u)^{3-k} \right\} 0 \le u \le 1$$

We can simplify this formula as follow:

$$\left. \begin{aligned} B(u) &= P_0(1-u)^3 + 3P_1u(1-u)^2 + 3P_2u^2(1-u) + P_3u^3 \\ B(u) &= u^3(-P_0+3P_1-3P_2+P_3)+u^2(3P_0-6P_1+3P_2)+u(-3P_0+3P_1)+P_0 \\ B(u) &= u^3(P_3+3(P_1-P_2)-P_0)+3u^2(P_0-2P_1+P_2)+3u(P_1-P_0)+P_0 \end{aligned} \right\} 0 \le u \le 1$$

Proof that P_C is the middle of the cubic curve

Let's calculates the position of P_C:

$$P_C = m(P_A, P_B)$$
$$P_C = m(m(P_{01}, P_{12}), m(P_{12}, P_{23}))$$
$$P_C = m(m(m(P_0, P_1), m(P_1, P_2)), m(m(P_1, P_2), m(P_2, P_3)))$$
$$P_C = m\left(m\left(\frac{P_0 + P_1}{2}, \frac{P_1 + P_2}{2}\right), m\left(\frac{P_1 + P_2}{2}, \frac{P_2 + P_3}{2}\right)\right)$$
$$P_C = m\left(\frac{P_0 + 2P_1 + P_2}{4}, \frac{P_1 + 2P_2 + P_3}{4}\right)$$
$$P_C = \frac{P_0 + 3P_1 + 3P_2 + P_3}{8}$$
$$P_C = \frac{P_0 + 3(P_1 + P_2) + P_3}{8}$$

Now Lets calculate the position of the cubic bezier at u=0.5:

$$B\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^3 (P_3 + 3(P_1 - P_2) - P_0) + 3\left(\frac{1}{2}\right)^2 (P_0 - 2P_1 + P_2) + 3\left(\frac{1}{2}\right)(P_1 - P_0) + P_0$$
$$B\left(\frac{1}{2}\right) = \frac{1}{8}(P_3 + 3(P_1 - P_2) - P_0) + \frac{6}{8}(P_0 - 2P_1 + P_2) + \frac{12}{8}(P_1 - P_0) + \frac{8}{8}P_0$$
$$B\left(\frac{1}{2}\right) = \frac{P_3 + 3P_1 - 3P_2 - P_0 + 6P_0 - 12P_1 + 6P_2 + 12P_1 - 12P_0 + 8P_0}{8}$$
$$B\left(\frac{1}{2}\right) = \frac{P_0 + 3P_1 + 3P_2 + P_3}{8}$$
$$B\left(\frac{1}{2}\right) = \frac{P_0 + 3(P_1 + P_2) + P_3}{8}$$

Conclusion:

$P_C = B\left(\frac{1}{2}\right)$ and therefore P_C is the middle of the cubic curve.

Proof that the first half matches exactly the main cubic

Let's call the curve determined by the four control points $\{P_0, P_{01}, P_A, P_C\}$ B_{sub1}.
Here is the formulas for each of the control points:

$$P_0 = P_0 \qquad P_A = \frac{P_0 + 2P_1 + P_2}{4}$$

$$P_{01} = \frac{P_0 + P_1}{2} \qquad P_C = \frac{P_0 + 3P_1 + 3P_2 + P_3}{8}$$

We have, for $0 \le u \le 1$:

$$B_{sub1}(u) = u^3\left(P_C + 3(P_{01} - P_A) - P_0\right) + 3u^2\left(P_0 - 2P_{01} + P_A\right) + 3u\left(P_{01} - P_0\right) + P_0$$

$$B_{sub1}(u) = u^3\left(\frac{P_0 + 3P_1 + 3P_2 + P_3}{8} + 3\left(\frac{P_0 + P_1}{2} - \frac{P_0 + 2P_1 + P_2}{4}\right) - P_0\right)$$
$$+ 3u^2\left(P_0 - 2\frac{P_0 + P_1}{2} + \frac{P_0 + 2P_1 + P_2}{4}\right) + 3u\left(\frac{P_0 + P_1}{2} - P_0\right) + P_0$$

$$B_{sub1}(u) = u^3\left(\frac{P_0 + 3P_1 + 3P_2 + P_3 + 12P_0 + 12P_1 - 6P_0 - 12P_1 - 6P_2 - 8P_0}{8}\right)$$
$$+ 3u^2\left(\frac{4P_0 - 4P_0 - 4P_1 + P_0 + 2P_1 + P_2}{4}\right) + 3u\left(\frac{P_0 + P_1 - 2P_0}{2}\right) + P_0$$

$$B_{sub1}(u) = u^3\left(\frac{P_3 + 3P_1 - 3P_2 - P_0}{8}\right) + 3u^2\left(\frac{P_0 - 2P_1 + P_2}{4}\right) + 3u\left(\frac{P_1 - P_0}{2}\right) + P_0$$

$$B_{sub1}(u) = \left(\frac{u}{2}\right)^3\left(P_3 + 3P_1 - 3P_2 - P_0\right) + 3\left(\frac{u}{2}\right)^2\left(P_0 - 2P_1 + P_2\right) + 3\left(\frac{u}{2}\right)\left(P_1 - P_0\right) + P_0$$

$$B_{sub1}(u) = B\left(\frac{u}{2}\right)$$

Since $B_{sub1}(u) = B\left(\frac{u}{2}\right)$ for $0 \le u \le 1$, B_{sub1} defined from 0 to 1 is the same as B from 0 to 0.5 since u and u/2 are continuous functions, B_{sub1} draws exactly the first half of the main cubic bezier.

Proof that the second half matches exactly the main cubic

Let's call the curve determined by the four control points $\{P_C, P_B, P_{23}, P_3\}$ B_{sub2}.
Here is the formulas for each of the control points:

$$P_C = \frac{P_0 + 3P_1 + 3P_2 + P_3}{8} \qquad P_{23} = \frac{P_2 + P_3}{2}$$

$$P_B = \frac{P_1 + 2P_2 + P_3}{4} \qquad P_3 = P_3$$

We have, for $0 \le u \le 1$:

$$B_{sub2}(u) = u^3(P_3 + 3(P_B - P_{23}) - P_C) + 3u^2(P_C - 2P_B + P_{23}) + 3u(P_B - P_C) + P_C$$

$$\begin{aligned}
B_{sub2}(u) = {} & u^3\left(P_3 + 3\left(\frac{P_1 + 2P_2 + P_3}{4} - \frac{P_2 + P_3}{2}\right) - \frac{P_0 + 3P_1 + 3P_2 + P_3}{8}\right) \\
& + 3u^2\left(\frac{P_0 + 3P_1 + 3P_2 + P_3}{8} - 2\frac{P_1 + 2P_2 + P_3}{4} + \frac{P_2 + P_3}{2}\right) \\
& + 3u\left(\frac{P_1 + 2P_2 + P_3}{4} - \frac{P_0 + 3P_1 + 3P_2 + P_3}{8}\right) \\
& + \frac{P_0 + 3P_1 + 3P_2 + P_3}{8}
\end{aligned}$$

$$\begin{aligned}
B_{sub2}(u) = {} & u^3\left(\frac{8P_3 + 6P_1 + 12P_2 + 6P_3 - 12P_2 - 12P_3 - P_0 - 3P_1 - 3P_2 - P_3}{8}\right) \\
& + 3u^2\left(\frac{P_0 + 3P_1 + 3P_2 + P_3 - 4P_1 - 8P_2 - 4P_3 + 4P_2 + 4P_3}{8}\right) \\
& + 3u\left(\frac{2P_1 + 4P_2 + 2P_3 - P_0 - 3P_1 - 3P_2 - P_3}{8}\right) \\
& + \frac{P_0 + 3P_1 + 3P_2 + P_3}{8}
\end{aligned}$$

$$\begin{aligned}
B_{sub2}(u) = {} & u^3\left(\frac{P_3 + 3P_1 - 3P_2 - P_0}{8}\right) + 3u^2\left(\frac{P_0 - P_1 - P_2 + P_3}{8}\right) \\
& + 3u\left(\frac{P_3 + P_2 - P_1 - P_0}{8}\right) + \frac{P_0 + 3P_1 + 3P_2 + P_3}{8}
\end{aligned}$$

OK at this point the formula above doesn't look familiar. The thing is that if we want to prove that B_{sub2} draws the second half of the main cubic, basically, $B_{sub2}(u)$ for u between 0 and 1 must represent B(v) for v between 0.5 and 1. Therefore what we want to prove is that $B_{sub2}(u) = B\left(\frac{1+u}{2}\right)$ for u between 0 and 1.

Let's calculate $B\left(\frac{1+u}{2}\right)$ for $0 \le u \le 1$

$$B\left(\frac{1+u}{2}\right) = \left(\frac{1+u}{2}\right)^3 (P_3 + 3(P_1 - P_2) - P_0) + 3\left(\frac{1+u}{2}\right)^2 (P_0 - 2P_1 + P_2) + 3\left(\frac{1+u}{2}\right)(P_1 - P_0) + P_0$$

$$B\left(\frac{1+u}{2}\right) = \left(\frac{1+3u+3u^2+u^3}{8}\right)(P_3 + 3(P_1 - P_2) - P_0) + 3\left(\frac{1+2u+u^2}{4}\right)(P_0 - 2P_1 + P_2)$$
$$+3\left(\frac{1+u}{2}\right)(P_1 - P_0) + P_0$$

$$B\left(\frac{1+u}{2}\right) = \frac{P_3+3P_1-3P_2-P_0}{8} + 3u\frac{P_3+3P_1-3P_2-P_0}{8} + 3u^2\frac{P_3+3P_1-3P_2-P_0}{8}$$
$$+u^3\frac{P_3+3P_1-3P_2-P_0}{8} + \frac{6P_0-12P_1+6P_2}{8} + 3u\frac{4P_0-8P_1+4P_2}{8} + 3u^2\frac{2P_0-4P_1+2P_2}{8}$$
$$+\frac{12P_1-12P_0}{8} + 3u\frac{4P_1-4P_0}{8} + \frac{8P_0}{8}$$

$$B\left(\frac{1+u}{2}\right) = u^3\left(\frac{P_3+3P_1-3P_2-P_0}{8}\right) + 3u^2\left(\frac{P_3+3P_1-3P_2-P_0}{8} + \frac{2P_0-4P_1+2P_2}{8}\right)$$
$$+3u\left(\frac{P_3+3P_1-3P_2-P_0}{8} + \frac{4P_0-8P_1+4P_2}{8} + \frac{4P_1-4P_0}{8}\right)$$
$$+\left(\frac{P_3+3P_1-3P_2-P_0}{8} + \frac{6P_0-12P_1+6P_2}{8} + \frac{12P_1-12P_0}{8} + \frac{8P_0}{8}\right)$$

$$B\left(\frac{1+u}{2}\right) = u^3\left(\frac{P_3+3P_1-3P_2-P_0}{8}\right) + 3u^2\left(\frac{P_0-P_1-P_2+P_3}{8}\right)$$
$$+3u\left(\frac{P_3+P_2-P_1-P_0}{8}\right) + \frac{P_0+3P_1+3P_2+P_3}{8}$$

$$B\left(\frac{1+u}{2}\right) = B_{sub2}(u)$$

Since $B_{sub2}(u) = B\left(\frac{1+u}{2}\right)$ for $0 \le u \le 1$, B_{sub2} defined from 0 to 1 is the same as B from 0.5 to 1 and since u and (1+u)/2 are continuous functions, B_{sub2} draws exactly the second half of the main cubic bezier.

Conclusion:

Any cubic bezier curve defined by four control points $\{P_0, P_1, P_2, P_3\}$ can be divided into two exact cubic halves.